Topology Qualifying Examination

M. Hedden

May 3, 2017

Instructions: Solve four out of the five problems. Even if you attempt more than four problems, indicate which problems you want graded. You must justify your claims either by direct arguments or by referring to theorems you know.

Problem 1. Give an example of two spaces that are homotopy equivalent but not homeomorphic. You must justify your answer.

Problem 2. Let F_n denote the free group on n generators. Use covering space theory to prove that all index two subgroups of F_2 are isomorphic to F_3.

Problem 3. There are natural inclusions $S^0 \subset S^1 \subset \cdots \subset S^n$, where each sphere includes into the next as the equator. With the usual CW cell structure of S^n (with just two cells) these subspheres are not subcomplexes.
(a) Put a cell structure on S^n such that each subsphere S^k is a subcomplex.
(b) The infinite dimensional sphere $S^\infty = \cup_n S^n$ is a cell complex as well. Compute the homology groups $H_*(S^\infty)$ (hint: use the CW decomposition from part (a)).

Problem 4. Let $X := \mathbb{RP}^2 \times \mathbb{RP}^2$. Find all the connected covering spaces of X. Justify your answer carefully. Which ones of these covering spaces are normal? Explain.

Problem 5. Let X be space obtained from the torus $T^2 = S^1 \times S^1$ and the unit disk D^2 by identifying the circle $S^1 \times \text{pt} \subset T^2$ with ∂D^2 by a homeomorphism (see figure on back). Calculate the homology groups of X.

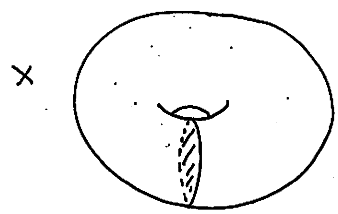

FIGURE 1. A torus and a disk